

Abstract

An inverse model is developed to predict internal tides using the Coupled-mode Shallow Water model (CSW), which includes realistic stratification, topography, and barotropic forcing. The inverse solution corresponds to finding the optimal forcing correction to CSW using a (direct or indirect) representer approach given a set of observations. Here, regional, $1/25^\circ$ -resolution solutions are obtained by fitting CSW to HRET M2 internal tides. The CSW inverse solution is consistent with HRET ($R^2=0.99$) and identifies forcing corrections due to poorly resolved generation at steep, shallow (<1000 m) bathymetry and small stratification (eigenspeed) errors in the deep ocean. Future work will focus on (1) parallelizing the model code so that it can run on basin scales, (2) directly fitting nadir altimetry to create a global map of the stationary internal tide, and (3) fitting SWOT data over each 21-day cycle to map the non-stationary internal tide.

The Coupled-mode Shallow Water model (CSW)

CSW (Kelly et al. 2021) evolves the first N_m baroclinic vertical modes (ϕ_n, c_n), forced by barotropic tides (F_n), damped by linear drag (r), and coupled by sloping topography (\mathbf{T}_{mn} ; Zaron et al. 2022):

$$\underbrace{\begin{bmatrix} \frac{(i\omega+r)H}{c_1^2} & 0 & (\nabla + \mathbf{T}_{11}) \cdot & \mathbf{T}_{21} \cdot \\ 0 & \frac{(i\omega+r)H}{c_2^2} & \mathbf{T}_{12} \cdot & (\nabla + \mathbf{T}_{22}) \cdot \\ (\nabla - \mathbf{T}_{11}) & -\mathbf{T}_{12} & \frac{(i\omega+r)}{H} + \frac{f}{H} \hat{k} \times & 0 \\ -\mathbf{T}_{21} & (\nabla - \mathbf{T}_{22}) & 0 & \frac{(i\omega+r)}{H} + \frac{f}{H} \hat{k} \times \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} p_1 \\ p_2 \\ \mathbf{U}_1 \\ \mathbf{U}_2 \end{bmatrix}}_{\mathbf{x}_F} = \underbrace{\begin{bmatrix} F_1 \\ F_2 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{b}}$$

$$\text{With } F_n = \mathbf{T}_{0n} \cdot \mathbf{U}_0 = -\mathbf{U}_0 \cdot \frac{\nabla H}{H} \phi_n |_{z=-H}, \text{ and } \mathbf{T}_{mn} = \begin{cases} -\frac{\nabla H}{H} \frac{1}{2} (1 - \phi_n^2) |_{z=-H} & n = m \\ -\frac{\nabla H}{H} \frac{c_m^2}{c_n^2} \phi_n \phi_m |_{z=-H} & n \neq m \end{cases}$$

Our inverse method follows Egbert et al. (1994), Egbert & Erofeeva (2002), and especially Egbert & Erofeeva (2014). A direct or indirect representer method minimizes the objective function (Bennett, 2005):

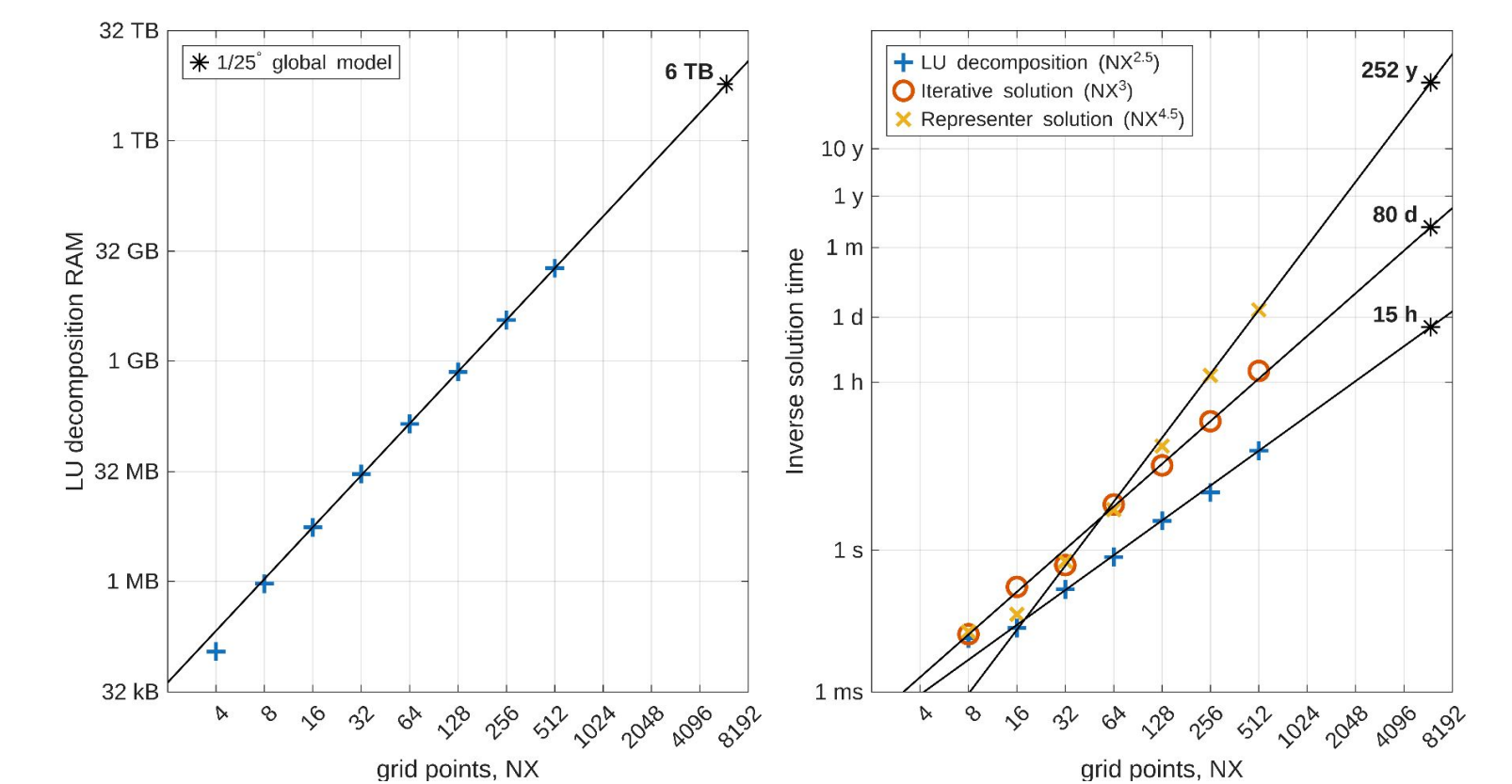
$$\mathcal{J}[\mathbf{x}] = \frac{1}{2} \int \left((\mathbf{A}\mathbf{x} - \mathbf{b})^T \mathbf{C}^{-1} (\mathbf{A}\mathbf{x} - \mathbf{b}) + (\mathbf{d} - \mathbf{L}\mathbf{x})^T \mathbf{C}_e^{-1} (\mathbf{d} - \mathbf{L}\mathbf{x}) \right) d\mathcal{A}$$

where \mathbf{L} is the observation operator, \mathbf{C}_e is the data-error covariance (assumed diagonal with $\sigma=0.2$ cm), and \mathbf{C} is the forcing-error covariance (with assumed scale $L_f = 5$ km and an amplitude based on $\pm 20\%$ errors in c^2).

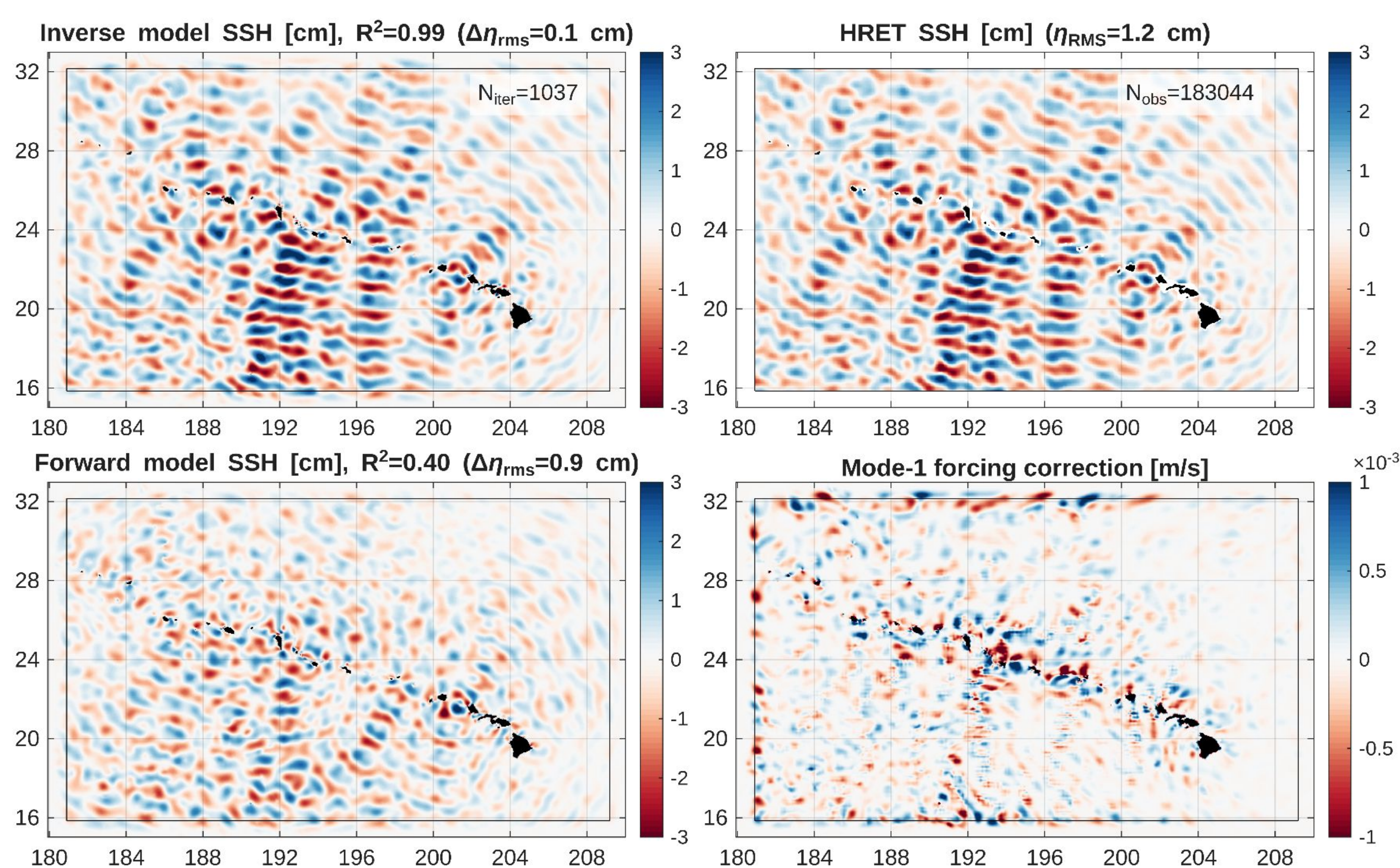
Implementation: Solutions for the M2 tide are run on a $1/25^\circ \approx 4$ km grid with $N_m=2$ modes and $r^{-1}=10$ days (although inverse solutions are insensitive to drag). The model uses WOA stratification, global bathymetry (topo_25.1), TPXO10 surface tides (Egbert & Erofeeva 2002), and HRET14 internal tide observations (Zaron & Elipot 2024).

\mathbf{A} is decomposed into LU components for rapid, repeated integration, as each representer or iteration requires an adjoint (backward) and forward model integration.

Below: Global solutions (*) require large RAM and CPU times.

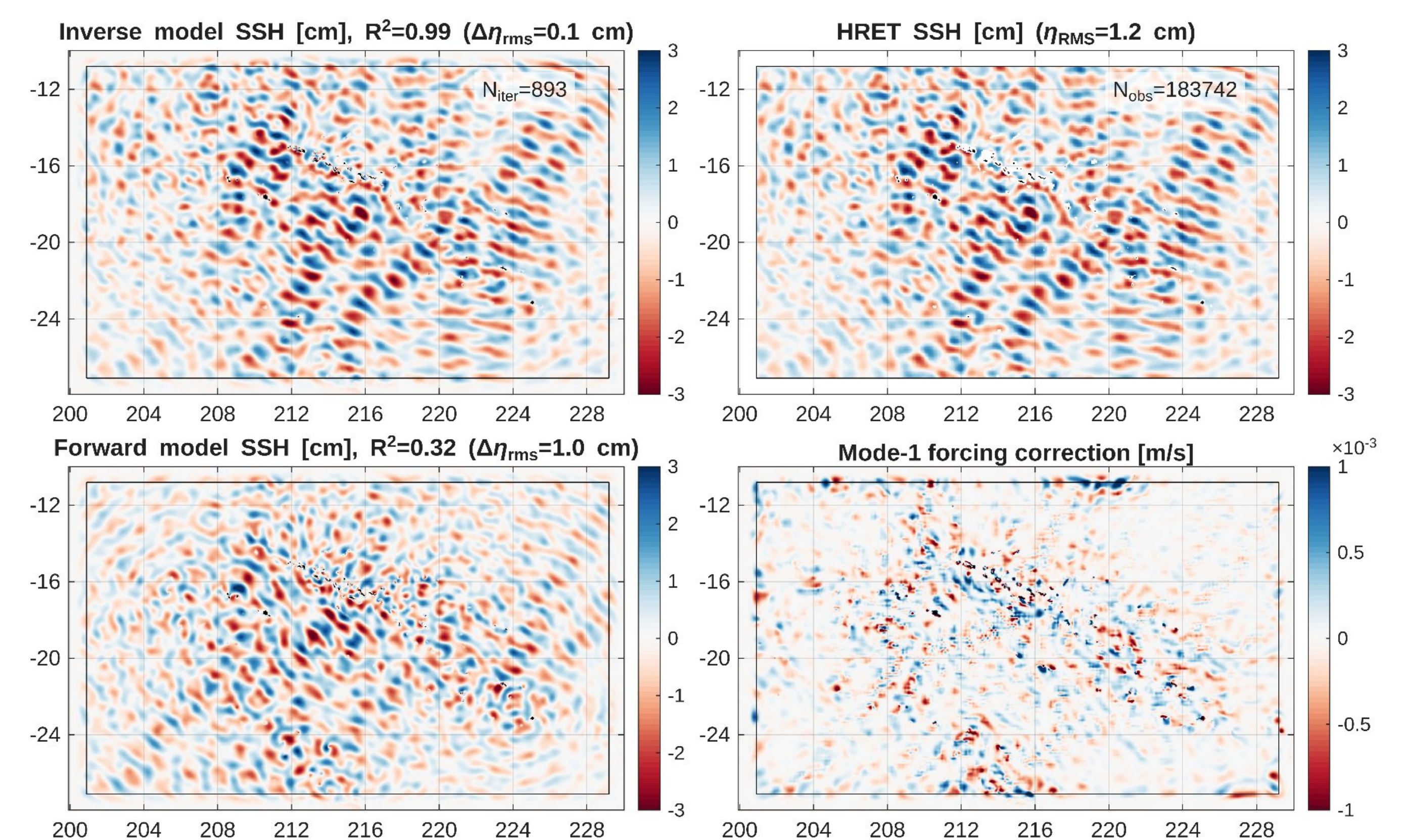


Hawai'i



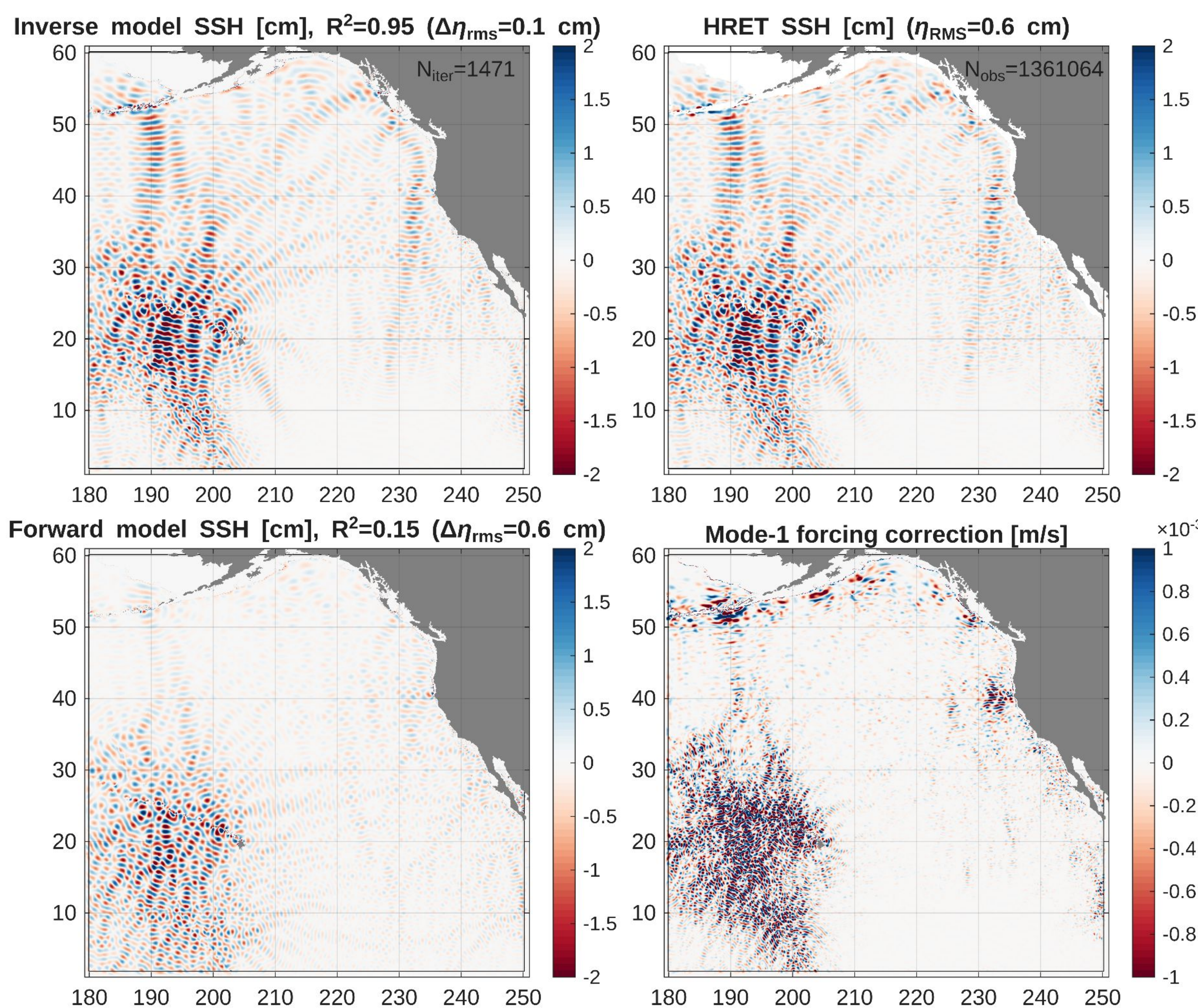
Above: The inverse CSW solution ($1/25^\circ$, $N_m=2$) explains 99% of HRET variance after 1037 iterations. The forward/prior model ($R^2=0.40$) is less energetic because barotropic forcing was excluded shallower than 1000 m. Forcing corrections are largest near islands and along the northern boundary, where remotely generated waves enter the domain.

French Polynesia



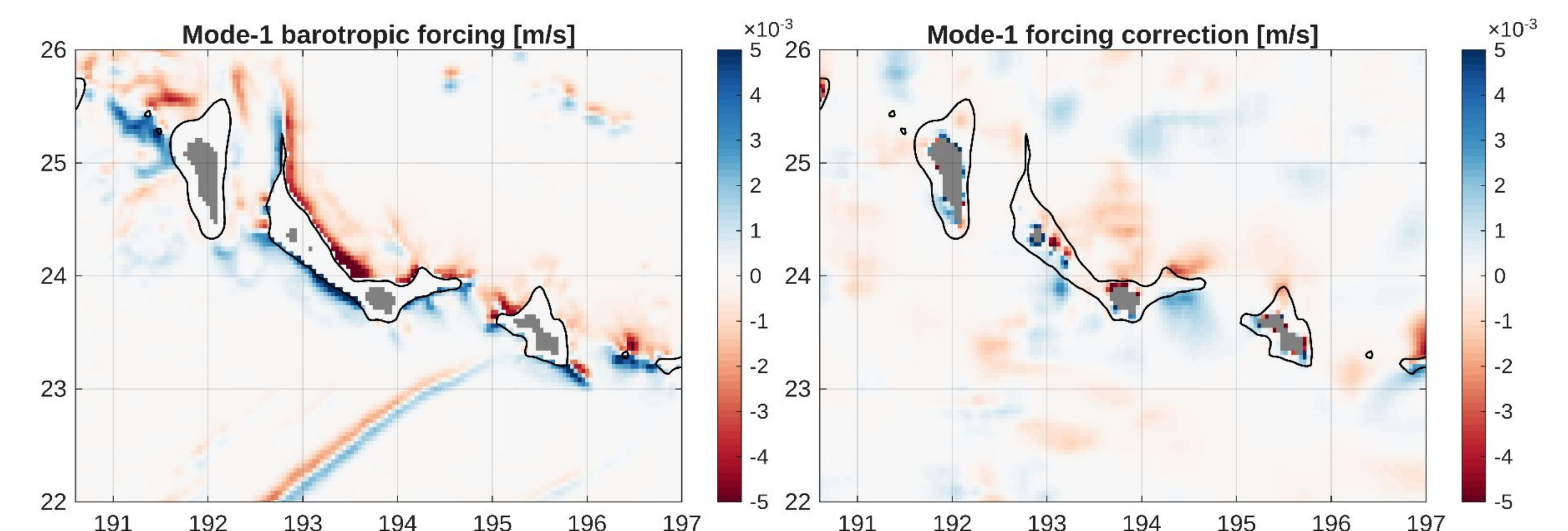
Above: As for Hawai'i, except French Polynesia. Internal tides in this region are more difficult to predict (forward/prior model $R^2=0.32$) than at the Hawaiian Ridge because of complicated barotropic tidal flow over steep, shallow bathymetry.

Northeast Pacific



Above: Inverse CSW solution ($1/25^\circ$, $N_m=1$) explains 95% of HRET variance in 1.36M observations after just 1471 iterations. The forward/prior solution ($R^2=0.15$) drastically underestimates generation at the Aleutian Islands and Mendocino Escarpment.

Detailed forcing at French Frigate Shoals, Hawai'i



Above: Barotropic flow over the Hawaiian Ridge generates internal tides in CSW. Forcing shallower than 1000 m (black contour) contains large errors and is removed from the forward/prior model to improve initial agreement with HRET. Forcing corrections from the inverse model reveal highly localized generation sources in shallow regions, likely due to strong barotropic flows over small-scale bathymetry. Diffuse corrections in deep water may be due to small errors in stratification (i.e., eigenspeed/wavelength).

Key Findings

1. A CSW inverse model with $1/25^\circ$ resolution and $N_m=2$ is consistent with HRET observations ($R^2=0.99$).
2. The CSW forward model has modest skill ($R^2 \approx 0.15-0.4$) when excluding forcing shallower than 1000 m. (Adding shallow forcing yields variable performance, $R^2 \approx \pm 0.6$).
3. Forcing corrections are associated with steep, small-scale, shallow bathymetry and deep-ocean stratification (eigenspeed) errors.
4. (Not shown) CSW inverse solutions are relatively insensitive to linear drag, r , the number of vertical modes, N_m , and the forcing-error length scale, L_f .
5. Iterative solutions converge much faster than the number of observations, indicating a reduced-dimensional data-space.

Next Steps

1. Directly assimilate nadir altimetry into CSW to map the stationary tide.
2. Parallelize the code for basin-scale simulations.
3. Assimilate SWOT data over each 21 day cycle to map the non-stationary tide.

Acknowledgments

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References

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